HL Paper 2

The population of mosquitoes in a specific area around a lake is controlled by pesticide. The rate of decrease of the number of mosquitoes is proportional to the number of mosquitoes at any time t. Given that the population decreases from 500 000 to 400 000 in a five year period, find the time it takes in years for the population of mosquitoes to decrease by half.

The acceleration in ms^{-2} of a particle moving in a straight line at time t seconds, $t \geqslant 0$, is given by the formula $a = -\frac{1}{2}v$. When $t = 0$, the velocity is $40\ \text{ms}^{-1}$.

Find an expression for v in terms of t .

An open glass is created by rotating the curve $y = x^2$, defined in the domain $x \in [0, 10]$, 2π radians about the y-axis. Units on the coordinate axes are defined to be in centimetres.

a. When the glass contains water to a height h cm, find the volume V of water in terms of h . [3]

b. If the water in the glass evaporates at the rate of 3 cm^3 per hour for each cm^2 of exposed surface area of the water, show that, [6]

$\frac{\mathrm{d}V}{\mathrm{d}t} = -3\sqrt{2\pi V}$, where t is measured in hours.

c. If the glass is filled completely, how long will it take for all the water to evaporate? [7]

Consider the differential equation $y\frac{\mathrm{d}y}{\mathrm{d}x} = \cos 2x$.

a. (i) Show that the function $y = \cos x + \sin x$ satisfies the differential equation. [10]

(ii) Find the general solution of the differential equation. Express your solution in the form $y = f(x)$, involving a constant of integration.

(iii) For which value of the constant of integration does your solution coincide with the function given in part (i)?

b. A different solution of the differential equation, satisfying $y = 2$ when $x = \frac{\pi}{4}$, defines a curve C. [12]

(i) Determine the equation of C in the form $y = g(x)$, and state the range of the function g.

A region R in the xy plane is bounded by C, the x-axis and the vertical lines $x = 0$ and $x = \frac{\pi}{2}$.

(ii) Find the area of R.

(iii) Find the volume generated when that part of R above the line $y = 1$ is rotated about the x-axis through 2π radians.

The acceleration of a car is $\frac{1}{40}(60 - v)$ ms^{-2}, when its velocity is v ms^{-2}. Given the car starts from rest, find the velocity of the car after 30 seconds.

(a) Solve the differential equation $\frac{\cos^2 x}{e^y} - e^{e^y}\frac{dy}{dx} = 0$, given that $y = 0$ when $x = \pi$.

(b) Find the value of y when $x = \frac{\pi}{2}$.

A. Prove by mathematical induction that, for $n \in \mathbb{Z}^+$, [8]

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}.$$

B. (a) Using integration by parts, show that $\int e^{2x}\sin x\,dx = \frac{1}{5}e^{2x}(2\sin x - \cos x) + C$. [17]

(b) Solve the differential equation $\frac{dy}{dx} = \sqrt{1 - y^2}e^{2x}\sin x$, given that $y = 0$ when $x = 0$,

writing your answer in the form $y = f(x)$.

(c) (i) Sketch the graph of $y = f(x)$, found in part (b), for $0 \leqslant x \leqslant 1.5$.

Determine the coordinates of the point P, the first positive intercept on the x-axis, and mark it on your sketch.

(ii) The region bounded by the graph of $y = f(x)$ and the x-axis, between the origin and P, is rotated 360° about the x-axis to form a solid of revolution.

Calculate the volume of this solid.

A particle moves in a straight line with velocity v metres per second. At any time t seconds, $0 \leqslant t < \frac{3\pi}{4}$, the velocity is given by the differential equation $\frac{dv}{dt} + v^2 + 1 = 0$.

It is also given that $v = 1$ when $t = 0$.

a. Find an expression for v in terms of t . [7]

b. Sketch the graph of v against t , clearly showing the coordinates of any intercepts, and the equations of any asymptotes. [3]

c. (i) Write down the time T at which the velocity is zero. [3]

(ii) Find the distance travelled in the interval [0, T] .

d. Find an expression for s , the displacement, in terms of t , given that $s = 0$ when $t = 0$. [5]

e. Hence, or otherwise, show that $s = \frac{1}{2}\ln\frac{2}{1+v^2}$. [4]
